Chapter 4: Blackbody radiation and the birth of quantum physics: Problems

Luis M. Molina

Departamento de Física Teórica, Atómica y Óptica

Quantum Physics

Problem 1.1

Question

Express Planck's formula for the density of energy $\rho_T(\nu)$ in a blackbody cavity in terms of the wavelenght λ. Then, prove Wien's displacement law, calculating Wien's constant.

Problem 1.1

Question

Express Planck's formula for the density of energy $\rho_T(\nu)$ in a blackbody cavity in terms of the wavelenght λ. Then, prove Wien's displacement law, calculating Wien's constant.

Solution

We start from:

$$\rho_T(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1} \tag{1}$$

Then, as $\nu = c/\lambda$, we have:

$$\rho_T(\lambda) = \rho_T(\nu)\left|\frac{d\nu}{d\lambda}\right| = \frac{8\pi h \cancel{c^3}}{\cancel{c^3}\lambda^3}\frac{c}{\lambda^2}\frac{1}{e^{hc\lambda kT}-1} = \boxed{\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}} \tag{2}$$

Problem 1.1

Solution (cont.)

Now, to find the wavelength λ_{max} for which $\rho_T(\lambda)$ reaches a maximum, we equal the derivative of the function to 0:

$$\frac{\partial \rho_T(\lambda)}{\partial \lambda} = 8\pi hc \left(\frac{hc}{kT\lambda^7} \frac{e^{hc/\lambda kT}}{\left(e^{hc/\lambda kT} - 1\right)^2} - \frac{1}{\lambda^6} \frac{5}{e^{hc/\lambda kT} - 1} \right) = 0$$
$$\Longrightarrow \quad \frac{hc}{\lambda kT} \frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT} - 1} - 5 = 0 \qquad (3)$$

Now taking $x = hc/\lambda kT$, we need to solve the equation $\dfrac{xe^x}{e^x - 1} = 5$.

This can only be solved numerically, with the result $x = 4{,}965114$. Then, finally:

$$\lambda_{max} T = \frac{hc}{xk} = \frac{6{,}626 \cdot 10^{-34} J \cdot s \cdot 2{,}9979 \cdot 10^8 m/s}{4{,}965114 \cdot 1{,}3806 \cdot 10^{-23} J/K} = 2{,}897 \cdot 10^{-3} m \cdot K \qquad (4)$$

Problem 1.2

Question

A cavity at 6000K has an energy distribution corresponding to a blackbody. We make a small hole in it with 1mm diameter. Calculate the power irradiated through the hole in the wavelength interval between 5500 and 5510 Å, as well as the total power

Problem 1.2

Question

A cavity at 6000K has an energy distribution corresponding to a blackbody. We make a small hole in it with 1mm diameter. Calculate the power irradiated through the hole in the wavelength interval between 5500 and 5510 Å, as well as the total power

Solution

As the wavelength interval is rather small, we can approximate the total backbody radiancy within that interval as:

$$R_T = \int_{\lambda_A}^{\lambda_B} \frac{c}{4} \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1} \approx \frac{2\pi hc^2}{\lambda_M^5 e^{hc/\lambda_M kT} - 1} \cdot (\Delta\lambda) \tag{5}$$

with $\Delta\lambda = \lambda_B - \lambda_A$ and $\lambda_M = (\lambda_B + \lambda_A)/2$

Then, the total power will be the radiancy times the area of the hole:

$$P = \frac{2\pi \cdot 6{,}626 \cdot 10^{-34} J \cdot s \cdot (3 \cdot 10^8) m^2/s^2}{(0{,}5505 \cdot 10^{-6} m)^5} \frac{1}{e^{4{,}359} - 1} \cdot 10^{-9} m \cdot (10^{-6} \cdot \pi m^2) = 0{,}301\ W$$

Problem 1.3

Solution (cont.)

For the total power, we use Stefan's law:

$$R_T = \sigma T^4 = 5{,}6704 \cdot 10^{-8}\, Js^{-1} m^{-2} K^4 \cdot (6000K)^4 = 7{,}3488 \cdot 10^7\ W/m^2$$

$$P = R_T \cdot A = 7{,}3488 \cdot 10^7\, W/m^2 \cdot (3{,}1416 \cdot 10^{-6} m^2) = 230{,}87\, W$$

Problem 1.3

Solution (cont.)

For the total power, we use Stefan's law:

$$R_T = \sigma T^4 = 5{,}6704 \cdot 10^{-8}\, Js^{-1} m^{-2} K^4 \cdot (6000K)^4 = 7{,}3488 \cdot 10^7 \; W/m^2$$

$$P = R_T \cdot A = 7{,}3488 \cdot 10^7\, W/m^2 \cdot (3{,}1416 \cdot 10^{-6} m^2) = 230{,}87\, W$$

Question

In a thermonuclear explosion the temperature in the fireball is momentarily $10^7 K$. Find the wavelength at which the radiation emitted is a maximum.

Problem 1.3

Solution (cont.)

For the total power, we use Stefan's law:

$$R_T = \sigma T^4 = 5{,}6704 \cdot 10^{-8}\, Js^{-1} m^{-2} K^4 \cdot (6000K)^4 = 7{,}3488 \cdot 10^7\ W/m^2$$
$$P = R_T \cdot A = 7{,}3488 \cdot 10^7\, W/m^2 \cdot (3{,}1416 \cdot 10^{-6} m^2) = 230{,}87\, W$$

Question

In a thermonuclear explosion the temperature in the fireball is momentarily $10^7 K$. Find the wavelength at which the radiation emitted is a maximum.

Solution

We apply Wien's law, obtaining:

$$\lambda_{max} = \frac{2{,}897 \cdot 10^{-3} m \cdot K}{10^7 K} = 2{,}897 \cdot 10^{-10} m = 2{,}897\, \text{Å}$$

which represent rather energetic X-rays.