Chapter 3: Planck's theory of blackbody radiation

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Quantum Physics

Oscillator statistics revisited

- The real problem in the derivation of the radiancy spectrum performed in the previous chapter lies in the $\bar{\epsilon} = kT$ assumption.
- While at small frequencies ($\nu \rightarrow 0$) the approximation $\bar{\epsilon} \approx kT$ provides correct results, in order to avoid the divergency at large values of ν, it must be fulfilled that $\bar{\epsilon} \xrightarrow[\nu \rightarrow \infty]{} 0$
- Then, the average energy $\bar{\epsilon}$ of the standing waves must be a function of ν satisfying the constraints described above.
- We will now analyze the origin of the equipartition law $\bar{\epsilon} = kT$. It can be derived from a important result of classical statistical physics called the **Boltzmann distribution**.
- When we have a large number of entities in thermal equilibrium at a given temperature, the probability $P(\epsilon)$ of finding a given entity with energy in the interval between ϵ and $\epsilon + d\epsilon$ is:

$$P(\epsilon) = \frac{e^{-\epsilon/kT}}{kT} \tag{1}$$

Oscillator statistics revisited

- We can now calculate the average energy of each entity by evaluating the expression:

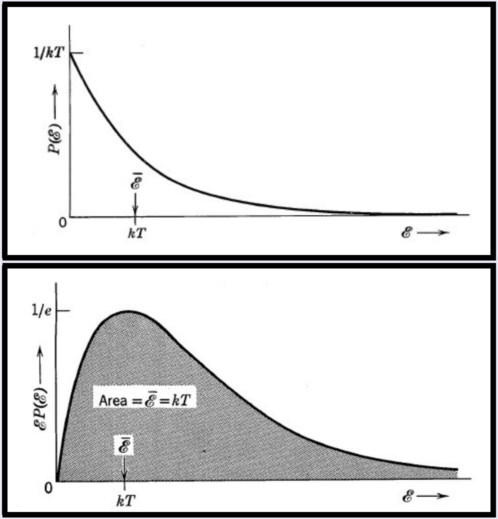
$$\bar{\epsilon} = \frac{\int_0^\infty \epsilon P(\epsilon) d\epsilon}{\int_0^\infty P(\epsilon) d\epsilon} = \frac{-(kT + \epsilon) e^{-\epsilon/kT} \Big|_0^\infty}{-e^{-\epsilon/kT} \Big|_0^\infty} = kT \qquad (2)$$

- In the previous discussion, we have assumed a continuous energy spectrum; but the formulas can be easily transformed to take into account the case of a discrete energy spectrum, by replacing integrals by summations over all the energy levels ϵ_n:

$$\bar{\epsilon} = \frac{\sum_0^\infty \epsilon_n P(\epsilon_n)}{\sum_0^\infty P(\epsilon_n)}, \quad \text{with} \quad P(\epsilon_n) = \frac{e^{-\epsilon_n/kT}}{kT} \qquad (3)$$

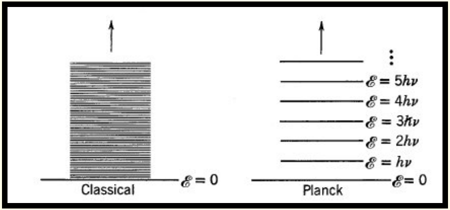
- The term in the numerator is the weighted sum of all the possible energies times the probability for an entity to be found with that energy, which decays exponentially with increasing energy.

Oscillator statistics revisited

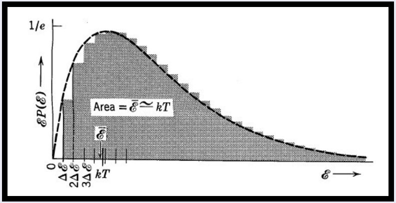


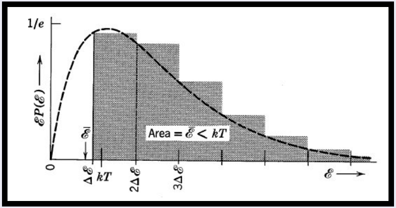
Planck's discretization hypothesis

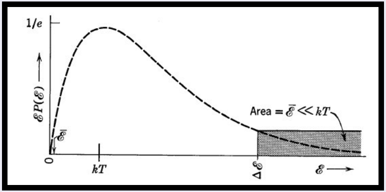
- The great contribution of Planck's to the solution of the problem was to realize that the correct limit of $\bar{\epsilon}$ at high frequencies, approaching zero, could be obtained by assuming that **the energy spectrum of standing waves within the blackbody cavity is not continuous, but discrete.**
- Then, in contradiction to the ideas of classical physics, he took for the energy spectrum only some discrete and **uniformly distributed** allowed values $\epsilon_n = 0, \Delta\epsilon, 2\Delta\epsilon, 3\Delta\epsilon, ...$



Planck's discretization hypothesis







- The average energy $\bar{\epsilon}$ monotonously decreases as the size of the energy interval $\Delta\epsilon$ increases with respecto the kT value.
- Then, Planck made the interval $\Delta\epsilon$ an increasing function of ν. Assuming these quantities to be proportional

$$\Delta\epsilon = h\nu \tag{4}$$

an excellent agreement with the experimental results is found.

Planck's formula for the average energy

- The value of the proportionality constant h (Planck's constant) was obtained by fitting the results to the existing experimental information for the blackbody radiation. It is now accepted to be $h = 6{,}626 \cdot 10^{-34} J \cdot s$, a very small value.
- We will now derive Planck's expression for the blackbody spectrum. Taking $\epsilon_n = nh\nu$, with $n = 0, 1, 2, \ldots$, and $P(\epsilon_n) = e^{(-\epsilon_n/kT)}/kT$ we obtain:

$$\bar{\epsilon}(\nu) = \frac{\sum\limits_{n=0}^{\infty} \frac{nh\nu}{kT} e^{-nh\nu/kT}}{\sum\limits_{n=0}^{\infty} \frac{1}{kT} e^{-nh\nu/kT}} = kT \frac{\sum\limits_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum\limits_{n=0}^{\infty} e^{-n\alpha}} = kT\left(-\alpha \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}\right) \tag{5}$$

  where we use the auxiliary parameter $\alpha = \frac{n\nu}{kT}$
- Now, using the fact that:

  $$1/(1-e^{-\alpha}) = 1 + e^{-\alpha} + e^{-2\alpha} + e^{-3\alpha} + \ldots = \sum_{n=0}^{\infty} e^{-n\alpha}$$
- We can finally calculate $\bar{\epsilon}(\nu)$:

Planck's formula for the average energy

$$\bar{\epsilon}(\nu) = -h\nu \frac{d}{d\alpha} \ln(1-e^{-\alpha})^{-1} = \frac{-h\nu}{(1-e^{-\alpha})^{-1}}(-e^{-\alpha})(1-e^{-\alpha})^{-2} =$$

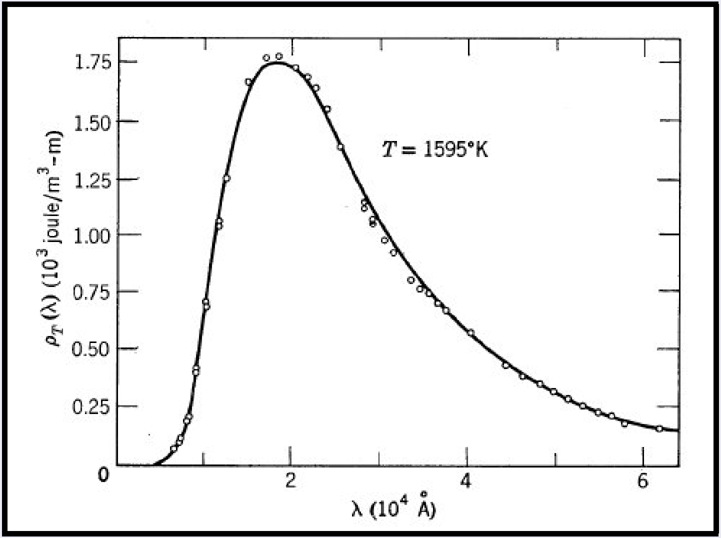
$$= \frac{h\nu e^{-\alpha}}{1-e^{-\alpha}} = \boxed{\frac{h\nu}{e^{h\nu/kT}-1}} \quad (6)$$

- As expected, $\bar{\epsilon}(\nu) \to kT$ in the limit of small frequencies, and $\bar{\epsilon}(\nu) \to 0$ for $\nu \to \infty$
- Then, we obtain the following formula for the energy density within the blackbody cavity:

$$\boxed{\rho_T(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT}-1} d\nu} \quad (7)$$

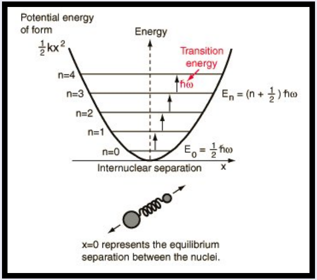
- The figure in the next slide shows the excelent agreement between theory (solid curve) and the experimental data (circles)

Planck's blackbody spectrum



Planck's postulate: physical implications

- We can summarize Planck's key idea for solving the blackbody problem as the following Postulate:
  Any physical quantity with one degree of freedom which executes simple harmonic oscillations can possess only total energies according to the relation:
  $$\epsilon = nh\nu \qquad n = 0, 1, 2, 3, ...$$
  where ν is the frequency of the oscillation, and h is the Planck's constant.
- If a system obbeys Planck's postulate, and therefore its energy spectrum in discrete, it is commonly stated that the energy of the system is **quantized**, with each of the allowed energy states being called **quantum states**.



- What is the origin of energy quantization of electromagnetic waves within the blackbody cavity? The answer is related to the emission processes taking place at the walls of the cavity. Radiation is emitted by atoms in the walls which undergo harmonic oscillations.
- We will later see that the spectrum of the harmonic oscillator (see Figure) is characterized by evenly spaced energy levels, with and energy d ifference of $h\nu$

Derivation of Stefan's law from Planck's formula

- We can easily prove Stefan's law from Planck's expression for the spectral radiancy:

$$R_T(\nu) = \frac{c}{4}\rho_T(\nu) = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}} - 1}. \tag{8}$$

- Then, we simply integrate the expresion for $R_T(\nu)$:

$$R_T = \frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}\, d\nu \tag{9}$$

- By changing the integration variable $u = \frac{h\nu}{kT}$, with $du = \frac{h}{kT}\, d\nu$, we obtain:

$$R_T = \frac{2\pi k^4}{c^2 h^3} T^4 \int_0^\infty \frac{u^3}{e^u - 1}\, du. \tag{10}$$

- The integral can be done by in a number of ways (contour integration, for example: check wikipedia article for Stefan's law) and its value is $\pi^4/15$. We therefore obtain Stefan's law on the form $R_T = \sigma T^4$, with $\sigma = \frac{2\pi^5 k^4}{15c^2 h^3} = 5{,}6704 \times 10^{-8}\,\mathrm{J\,s^{-1}m^{-2}K^{-4}}$

Macroscopic systems

- Why do we never observe the discreteness of the energy transfers in a classical oscillator?
- Consider a point particle of 1 gr which is at one end of a string whose length is 1 meter and it is oscillating in the gravitational field. Assume that the angle corresponding to the maximum separation from the vertical is one degree. How many quanta are needed to start the motion of the pendulum?
- As we have a frequency $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = 0{,}498 s^{-1}$ and $x = 1{,}523 \cdot 10^{-4} m$, the potential energy E of the pendulum is $E = mgx = 1{,}492 \cdot 10^{-6} J$. Applying the quantum equation $E = nh\nu$, since in this case $h\nu = 3{,}301 \cdot 10^{-34} J$, we find the numer of quanta to be $n = 4{,}5 \cdot 10^{27}$!!!
- What is the moral of the story? To measure the discrete character of energy in a macroscopic system like that simple pendulum, we need to measure energies with an incredible precision of around 1 part in 10^{25}. This explains why quantum effects are completely unobservable at the macroscopic level.
- The tiny value of the Planck's constant h makes the graininess in the energy too fine to be distinguished from an energy continuum in macroscopic (i.e, classical) systems. Only when we face systems where ν is very large and/or $h\nu$ is of the order of ϵ, we are able to test Planck's postulate and observe quantum effects.